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The Unitary Equivalence Problem.

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ABSTRACT

The purpose of this dissertation is to establish a set of invariants for determining when two normal operators on a separable Hilbert space are unitarily equivalent.

In the first chapter we give some basic definitions and establish some of the notation which will be used in later chapters. We also state, without proof, a few of the basic results relating to commutative, weakly closed, symmetric rings with identity. We end the first chapter with a discussion of rings which are generated by operators of the form A_f, where f is a continuous function and for each g in the Hilbert Space $L^2[0,1]$, $A_f g = fg$.

In the second chapter we define the decomposition theory of Pedersen and show through examples that this decomposition theory is very well suited for the unitary equivalence problem. In this chapter, we also

characterize the maximal ideal space of a weakly closed, commutative, and symmetric ring which is generated by a normal operator on a separable Hilbert space. This characterization is in terms of ultrafilters of sets of complex numbers.

In Chapter III, with the aid of the decomposition theory of Pedersen, we associate with each normal operator on a separable Hilbert space a countable collection of measures and show that two normal operators are unitarily equivalent if and only if their associated measures are mutually absolutely continuous. We also compare this theory with the multiplicity theory of Porcelli and Butts.

CHAPTER I

INTRODUCTION

In this paper we are primarily concerned with the development of a set of invariants for determining when two normal operators on a separable Hilbert space are unitarily equivalent. For each normal operator we will construct a countable collection of measures and will show that two normal operators are unitarily equivalent if and only if their associated measures are mutually absolutely continuous. We begin with a collection of basic definitions and notation, which by no means will be complete. For a comprehensive discussion we suggest the works of Naimark [4] and Pedersen [6].

By H we will mean a complex Hilbert space, and if $x, y \in H$, then (x, y) and $\|x\|$ will be used to denote the inner product and norm respectively.

All of the operators considered will be bounded linear operators and therefore the word ''operator'' will be used to mean a bounded linear operator. By $B(H)$ we will mean the set of all operators on a Hilbert space H. If $A \in B(H)$, then A^* will denote the operator which is the adjoint of A.

If M is a ring contained in $B(H)$, then we say that M is symmetric provided that $A^* \in M$ whenever $A \in M$. Suppose that S is a set contained in $B(H)$, then S' will be $\{A \in B(H) \mid$ for each $B \in S$, $AB = BA$ and $AB^* = B^*A\}$. The collection S' which is called the commutant of S, is a weakly closed, symmetric ring which contains the identity operator. Moreover, if M is a weakly closed symmetric ring, containing the identity operator, then $M = (M')'$.

If M is a ring contained in $B(H)$ and $\xi \in H$, then by $M\xi$ we shall mean the closure of the linear manifold $\{A\xi \mid A \in M\}$. If $H = M\xi$ we shall say that ξ is a cyclic vector for M. If H is separable and M is a commutative ring in $B(H)$, then M', the commutant of M, has a cyclic vector. If M is also symmetric and weakly closed, then a necessary and sufficient condition for M to have a cyclic vector is that M be maximally commutative.

We will also make extensive use of the Gelfand-Naimark theorem in [4] which says that a commutative, symmetric Banach ring $M \subset B(H)$, containing the identity, is isometrically isomorphic to the ring of all complex valued continuous functions on the maximal ideal space of M. This isomorphism is called the Gelfand transform. The transform of an operator A in M will be denoted by $A^{\wedge}$.

Suppose that X is a locally compact topological space. A spectral measure on X is a function $P(\lambda)$ whose domain is the collection of all Borel subsets of X and whose values are projections in a fixed Hilbert space H, where 1) $P(X) = 1$ and 2) for arbitrary $x \in H$, the set function μ, which is defined for every Borel set E by $\mu(E) = (P(E)x,x)$, is countably additive. It is easily seen that if $P(\lambda)$ is a spectral measure, then it is modular and multiplicative; that is, if E and F are Borel subsets of X, then $P(E \cup F) + P(E \cap F) = P(E) + P(F)$ and $P(E \cap F) = P(E)P(F)$. For f a complex valued Borel function on X, let $\|f\|_P$ be the essential supremum of $|f|$ with respect to $P(\lambda)$; that is, for an arbitrary nonnegative real number λ, E a Borel set with $P(E) = 0$, we have, that $|f(x)| \leq \lambda$ for $x \notin E$, if and only if $\|f\|_P \leq \lambda$. The powerful connection between spectral measures and normal operators is given by the following theorem.

Theorem 1.1. If A is a normal operator on a separable Hilbert space H, then there exists a compact, set T of the complex plane, and a spectral measure $P(\lambda)$ on T.

This spectral measure is called the spectral resolution for A and has the following properties:

i) $A = \int \lambda dP(\lambda)$;

ii) $B \in (\{A\}')'$ if and only if there exists a bounded Borel function f such that

$B = f(A) = \int f(\lambda)dP(\lambda)$ and $\|B\| = \|f\|_P$; and

iii) if $\{f_n\}$ is a bounded sequence of Borel functions and if $f(x) = \lim_{n\to\infty} f_n(x)$ except on a Borel set E with $P(E) = 0$, then $\{f_n(A)\}$ converges to $f(A)$ in the strong operator topology.

The proof of i) can be found in any text which contains the spectral theorem, for example [4] and [7], ii) is proven in [7] for self-adjoint operators but is easily extended to normal operators, and iii) follows directly from Lebesque's dominated convergence theorem.

We will consider $L^\infty[0,1]$ (with respect to Lebesque measure) to be a ring of operators acting on the Hilbert space $L^2[0,1]$ (also with respect to Lebesque measure) by multiplication; more explicitly, for $f \in L^\infty[0,1]$ we define the operator $A_f \in B(L^2[0,1])$ by $A_f g = fg$ for all $g \in L^2[0,1]$. The elements of this ring are the primary source for many examples in this paper. The following lemma and theorem give us some properties of these operators which will be used in later chapters.

Lemma 1.1. Suppose f is a continuous complex valued function on $[0,1]$. For each Borel set E of complex numbers, let $g_E(x) = \Pi_{f^{-1}(E)}(x)$, where $\Pi_{f^{-1}(E)}$ is the characteristic function of $f^{-1}(E)$. If $P(\lambda)$ is the spectral resolution of A_f, then $A_{g_E} = P(E)$.

Proof: Let $\epsilon > 0$ and let $\{S_i\}_{i=1}^n$ be a collection of Borel sets which cover the range of f and has the additional properties: 1) $S_i \cap S_j = \emptyset$ for $i \neq j$, and 2) $x,y \in S_i$ implies $|x - y| < \epsilon$ for $i = 1,2,\cdots,n$. Let $\lambda_i' \in S_i$ and let $E_{\lambda_i} = f^{-1}(S_i)$. If $x \in E_{\lambda_i}$, then

$$|f(x) - \sum_{i=1}^{n} \lambda_i' \, g_{E_i}(x)| = |f(x) - \lambda_i'| < \epsilon.$$

Hence,

$$\|A_f - \Sigma_{i=1}^{n} \lambda_i' \, A_{g_{E_i}}\| < \epsilon.$$

Therefore, by the uniqueness of spectral measure $A_{g_E} = P(E)$.

By using the above, in the following theorem we characterize the weakly closed, symmetric, commutative ring generated by the operator A_f, where f is a continuous function on $[0,1]$.

Theorem 1.2. Let f be a continuous, complex valued function on $[0,1]$. Then $(\{A_f\}')' = \{A_{g' \circ f} | g'$ is a bounded complex valued Borel function on the range of $f\}$.

Proof: Let B be a positive definite operator in $(\{A_f\}')'$. Since $(\{A_f\}')'$ is contained in $L^\infty[0,1]$, there exists a $g \in L^\infty[0,1]$ such that $B = A_g$. By Theorem 1.1 there exists a bounded Borel function g' such that

$$B = \int g'(\lambda)dP(\lambda),$$

where $P(\lambda)$ is the spectral resolution of A_f. There exists a sequence of simple functions $\{S_n\}$ such that $S_n \to g'$ uniformly on the range of f. It follows that $S_n \circ f \to g' \circ f$ uniformly on $[0,1]$. Suppose that $S_n = \Sigma_{i=1}^{k_n} a_{n_i} \Pi_{E_{n_i}}$. Then for $x, y \in L^2[0,1]$,

$$\int S_n(\lambda)d(P(\lambda)x,y) = \int \Sigma_{i=1}^{k_n} a_{n_i} \Pi_{E_{n_i}}(\lambda)d(P(\lambda)x,y)$$

$$= \Sigma_{i=1}^{k_n} a_{n_i}(P(E_{n_i})x,y)$$

$$= \Sigma_{i=1}^{k_n} a_{n_i}(A_{\Pi_{f^{-1}(E_{n_i})}}x,y)$$

$$= \int_0^1 \Sigma_{i=1}^{k_n} a_{n_i} \Pi_{f^{-1}(E_{n_i})}(t)x(t)\overline{y(t)}dt$$

$$= \int_0^1 s_n(f(t))x(t)\overline{y(t)}dt.$$

Let $\epsilon > 0$ be given. Then there exists a positive number N such that for $n > N$ we have that $|g'(t) - s_n(t)| < \epsilon$ and $|g'(f(x)) - s_n(f(x))| < \epsilon$. Therefore, for $n > N$ we have that

$$\begin{aligned} |(A_g x,y) - (A_{g'\circ f}x,y)| &= |(A_g x,y) - \int s_n(\lambda)d(P(\lambda)x,y) \\ &\quad + \int_0^1 s_n(f(t))x(t)\bar{y}(t)dt - (A_{g'\circ f}x,y)| \\ &\leq |\int [g'(\lambda) - s_n(\lambda)]d(P(\lambda)x,y)| \\ &\quad + |\int_0^1 [s_n(f(x)) - g'(f(x))]x(t)\bar{y}(t)dt| \\ &< 2\,\epsilon\,\|x\|\,\|y\|. \end{aligned}$$

Hence, $B = A_g = A_{g'\circ f}$ and since any normal operator can be written as the sum of four positive definite operators we have established the lemma.

CHAPTER II

In this chapter we investigate the decomposition of a Hilbert space into a sum of subspaces each of which is cyclic for the ring in question. These cyclic subspaces will be the basis for determining when two normal operators are unitarily equivalent.

Throughout this section we will assume that H is a Hilbert space, M is a symmetric, weakly closed, commutative subring of $B(H)$ and that M', its commutant, has a cyclic vector ξ_0. It has been shown in [6] that under these conditions $\mathfrak{M}$, the maximal ideal space of M, is extremely disconnected; that is, the closures of open sets are open. If U is a clopen (open and closed) set in $\mathfrak{M}$, then Π_U, the characteristic function of U, is continuous. Hence there exists a projection in M such that its Gelfand transform is equal to Π_U; we denote this projection by P_U.

If $\eta \in H$, then there exists a measure γ on $\mathfrak{M}$ such that

$$(A\eta,\eta) = \int_{\mathfrak{M}} A^{\wedge}(m)d\gamma(m).$$

Moreover, γ has the property that if S is a measurable subset of $\mathfrak{M}$, then there exists a clopen set U such that $\gamma(S\Delta U) = 0$, where $S\Delta U$ is the symmetric difference $(S \setminus U) \cup (U \setminus S)$. The measure μ which is induced in the above way by ξ_0 has the additional property that its support is all of $\mathfrak{M}$.

For each $\eta \in H$ there exists $\varphi_\eta \in L'[\mathfrak{M},\mu]$ such that

$$(A\eta,\eta) = \int_{\mathfrak{M}} A^{\wedge}(m)\varphi_\eta(m)d\mu(m).$$

We denote by S_η the closure of $\{m \in \mathfrak{M} | \varphi_\eta(m) \neq 0\}$, S_η is a clopen set so there exists a projection P_{S_η} such that $P^{\wedge}_{S_\eta} = \Pi_{S_\eta}$.

With the above definitions we are now ready to define the decomposition which Pedersen, in [6], has shown will always exist under the stated conditions on M and M'.

<u>Definition 2.1.</u> Suppose M is a weakly closed, symmetric, and commutative subring of $B(H)$ and that M', its commutant, has a cyclic vector ξ_0. A canonical decomposition system for M is a collection $\{(K_\alpha, \eta_\alpha)\}_{\alpha\in\Gamma}$ such that $\eta_\alpha \in K_\alpha \subset H$ and

i) Γ is a well ordered set;

ii) $K_\alpha = M\eta_\alpha$, $H = \Sigma K_\alpha$ and $K_\alpha \perp K_\beta$ for $\alpha \neq \beta$;

iii) $\varphi_{\eta_\alpha} = \Pi_{S_{\eta_\alpha}}$; and

iv) $\alpha < \beta$ implies $S_{\eta_\beta} \subset S_{\eta_\alpha}$.

The sets S_{η_α} can be characterized independently of the η_α and ξ_0 by means of a local dimension function.

<u>Definition 2.2.</u> The dimension of H relative to M, which will be denoted by $\dim_M H$, is defined to be the smallest cardinal number c such that $H = \Sigma_{\alpha\in\Lambda} H\alpha$, where card $\Lambda = c$ and each H_α is a non-trivial cyclic subspace for M. For each clopen set $V \subset \mathfrak{M}$, we set $E_V = \{AP_V | A \in M\}$, and for each $m \in \mathfrak{M}$ we set $d(m) = \inf_V \{\dim_{E_V} PH | m \in V \subset \mathfrak{M}\}$. Then $d(m)$ is called the local dimension function.

A connection between the local dimension function and the S_{η_α} is given by the following theorem, the proof of which can be found in [6].

<u>Theorem 2.1.</u> Suppose $M, M', \xi_0, \{S_{\eta_\alpha}\}_{\alpha\in\Gamma}$ satisfy the conditions of Definition 2.1, and suppose H is separable, n is a positive integer and $m \in \mathfrak{M}$. Then $m \in S_{\eta_n}$ if and only if $d(m) \geq n$.

Before considering examples of the canonical decomposition we want to characterize the maximal ideal space of a weakly closed ring generated by a normal operator on a separable Hilbert space.

Let N be a normal operator on a separable Hilbert space H, and let $P(\lambda)$ be the spectral resolution for N; that is, $N = \int \lambda dP(\lambda)$. Also let Σ_N be the collection of all Borel subsets E of the plane such that $P(E) \neq 0$, and let $\mathfrak{M}_N$ be the collection of all ultrafilters of sets in Σ_N. For each E in Σ_N, put

$$\hat{E} = \{m \in \mathfrak{M}_N | E \in m\}.$$

<u>Theorem 2.2.</u> Let N be a normal operator on a separable Hilbert space H. Then $\mathfrak{M}_N$, with the topology induced by sets of the form $\hat{E}$, is the maximal ideal space of the weakly closed ring generated by N and N^*. In addition $P(E)^{\wedge}$, the Gelfand transform of $P(E)$, is $\Pi_{\hat{E}}$.

<u>Proof:</u> Let M be the weakly closed ring generated by N and N^*, and suppose also that $\mathfrak{M}$ is the collection of all multiplicative linear functionals on M, with the weak-* topology. Define Γ from $\mathfrak{M}$ into $\mathfrak{M}_N$ by the following. For each $\phi \in \mathfrak{M}$, put

$$\Gamma(\phi) = \{E \in \Sigma_N | \phi(P(E)) = 1\}.$$

We will prove that Γ is a homeomorphism. First we must show that $\Gamma(\phi)$ is indeed an ultrafilter. If $E,F \in \Gamma(\phi)$, then

$$\phi(P(E \cap F)) = \phi(P(E)P(F)) = \phi(P(E))\phi(P(F)) = 1$$

and $E \cap F \in \Gamma(\phi)$. Now $E \subset F$ and $E \in \Gamma(\phi)$, then

$$\begin{aligned}\phi(P(F)) = \phi(P((F \setminus E) \cup E) &= \phi(P(F \setminus E)) + \phi(P(E)) \\ &= \phi(P(F \setminus E)) + 1 \geq 1,\end{aligned}$$

and since $P(E)$ is a projection we must have equality, so that $F \in \Gamma(\phi)$. Finally assume that $E \cup F \in \Gamma(\phi)$. Then

$$1 = \phi(P(E \cup F)) = \phi(P(E) + P(F) - P(E \cap F))$$
$$= \phi(P(E)) + \phi(P(F)) - \phi(P(E \cap F)).$$

If both E and F were not in $\Gamma(\phi)$ we would have that $\phi(P(E)) = \phi(P(F)) = 0$ and hence $\phi(P(E \cap F)) = -1$, which is clearly impossible. Thus E **or** F is in $\Gamma(\phi)$, and therefore, $\Gamma(\phi)$ is an ultrafilter.

Now we shall show that Γ is onto. Let $m \in \mathfrak{M}_N$ and put

$$\phi_m(P(E)) = \begin{cases} 1 & \text{if } E \in \mathfrak{m}, \\ 0 & \text{otherwise} . \end{cases}$$

Since m is an ultrafilter, $E \cap F \in m$ if and only if E and F are both in m, and for $E \cap F = \phi$, $E \cup F \in m$ if and only if $E \in m$ or $F \in m$ but not both. The above implies that

1) $\phi_m(P(E \cap F)) = \phi_m(P(E))\phi_m(P(F))$; and

2) $\phi_m(P(E \cup F)) = \phi_m(P(E)) + \phi_m(P(F))$ if $E \cap F = \phi$.

We want to extend ϕ_m to a multiplicative linear functional on M. It can be extended linearly to operators of the form $\Sigma_{i=1}^{n} a_i P(E_i)$ and by 1) and 2) it will be linear and multiplicative on this class of operators. Moreover, if $S = \Sigma_{i=1}^{n} a_i P(E_i)$, and $E_i \cap E_j = \emptyset$ for $i \neq j$, then

$$3) \quad |\phi_m(S)| = |\Sigma_{i=1}^{n} a_i \phi_m(P(E_i))| \leq \max\{a_i\}_{i=1}^{n}$$
$$\leq \|S\|.$$

Let B be a positive definite operator in M. Then by Theorem 1.1 there exists a non-negative bounded Borel function f, on the spectrum of N, such that

$$B = \int f(\lambda) dP(\lambda)$$

and $\|f\|_P = \|B\|$. Let $\{S_n\}$ be an increasing sequence of simple functions which converge to f uniformly. If $S_n = \Sigma_{i=1}^{m_n} a_{n,i} \chi_{E_{n,i}}$, then let $S'_n = \Sigma_{i=1}^{m_n} a_{n,i} P(E_{n,i})$ and define

$$\bar{\phi}_m(B) = \lim_{n \to \infty} \phi_m(S'_n).$$

The above limit clearly exists and is unique by 3) and

Theorem 1.1. It is also easily seen by 1) and 2) that $\bar{\emptyset}_m \in \mathfrak{M}$. We now have that Γ is onto, since

$$\begin{aligned}\Gamma(\bar{\emptyset}_m) &= \{E \in \Sigma_N \mid \bar{\emptyset}(P(E)) = 1\} \\ &= \{E \in \Sigma_N \mid E \in m\} \\ &= m.\end{aligned}$$

If $\emptyset_1, \emptyset_2 \in \mathfrak{M}$ and $\Gamma(\emptyset_1) = \Gamma(\emptyset_2)$, then $\emptyset_1(P(E)) = \emptyset_2(P(E))$ for all $E \in \Sigma_N$. Hence by continuity they are equal and Γ is one to one. Since $\mathfrak{M}$ is compact and Hausdorff, all that remains to be shown is that Γ is continuous.

Let $\emptyset_0 \in \Gamma^{-1}(\hat{E})$ and let

$$U = \{\emptyset \in \mathfrak{M} \mid |\emptyset_0(P(E)) - \emptyset(P(E))| < 1/2\}.$$

Since $\Gamma(\emptyset_0) \in \hat{E}$, we have that $\emptyset_0(P(E)) = 1$. Hence if $\emptyset \in U$, $\emptyset(P(E))$ must be equal to 1 so that $\emptyset \in \Gamma^{-1}(\hat{E})$. Thus we have shown that $U \subset \Gamma^{-1}(\hat{E})$ and that Γ is continuous. Therefore, Γ is a homeomorphism.

<u>Remark 2.1.</u> If we consider $L^\infty[0,1]$ as a ring of oper-

ators acting on the Hilbert space $L^2[0,1]$ by multiplication and let N be multiplication by x, then $P(E) = 0$ if and only if E is a set of measure zero. Hence, by Theorem 2.2, the maximal ideal space of $L^\infty[0,1]$, the weakly closed ring generated by N, is the collection of all ultrafilters of sets of positive measure. Also by Theorem 2.2 and the results in Chapter I, if f is a real valued continuous function on $[0,1]$, and N is multiplication by f, then the maximal ideal space of the weakly closed ring generated by N is the collection of all ultrafilters of Borel sets with the property that $E = f^{-1}(f(E))$ and $\mu(E) > 0$.

In order to motivate the role played by the canonical decomposition in determining when two normal operators are unitarily equivalent, we shall exhibit the canonical decomposition for some special rings. In particular if $f \in L^\infty[0,1]$, we will consider the weakly closed ring generated by the operator A_f acting on $L^2[0,1]$ by multiplication by f. We begin with the following lemmas.

Lemma 2.1. Let X_1, X_2 be measurable subsets of $[0,1]$

and let g be a continuous, one to one function from X_1 onto X_2 with the additional property that $\mu(E) = 0$ if and only if $\mu(g(E)) = 0$, where μ is Lebesque measure on $[0,1]$. Then there exists $\phi_g \in L^1(X_1)$ such that:

i) $\int_{X_1} f(g(t))\phi_g(t)d\mu = \int_{X_2} f(t)d\mu$ for all $f \in L^\infty[X_2]$;

ii) $\int_{X_2} \phi_g(t)d\mu = \mu(X_2)$; and

iii) if ϕ_{g-1} is the $L^1[X_2]$ function that corresponds to g^{-1}, then $\phi_g(g^{-1}(t))\phi_{g-1}(t) = 1$ a.e.

<u>Proof</u>: Let $\mu_g(E) = \mu(g(E))$ for E a measurable subset of $[0,1]$ and let F be a measurable subset of X_2. Then

$$\begin{aligned}\int_{X_1} \Pi_F(g(t))d\mu_g &= \int_{X_1} \Pi_{g^{-1}(F)}(t)d\mu_g \\ &= \mu_g(g^{-1}(F)) \\ &= \mu(F) \\ &= \int_{X_2} \Pi_F(t)d\mu \ .\end{aligned}$$

Thus if S is a simple function on X_2, it follows that

$$\int_{X_1} S(g(t))d\mu_g = \int_{X_2} S(t)d\mu.$$

If $f \in L^\infty[X_2]$, there exists a sequence of simple functions $\{S_n\}$, which converge uniformly to f. Then $\{S_n \circ g\}$ will converge uniformly to $f \circ g$ and we have that

$$1)\quad \int_{X_1} f(g(t))d\mu_g = \int_{X_2} f(t)d\mu \quad \text{for all} \quad f \in L^\infty[X_2].$$

Since g takes sets of measure zero to sets of measure zero. μ_g and μ are mutually absolutely continuous. Therefore, there exists $\phi_g \in L^1[X_1]$, such that

$$2)\quad \int_{X_1} f(g(t))\phi_g(t)d\mu = \int_{X_1} f(g(t))d\mu_g$$

for all $f \in L^\infty[X_2]$. Combining 1) and 2) we have established i).

Now suppose $f \in L^\infty[X_2]$. Then

$$\int_{X_2} f(t)\phi_g(g^{-1}(t))\phi_{g-1}(t)d\mu$$

$$= \int_{X_2} f(g(g^{-1}(t)))\phi_g(g^{-1}(t))\phi_{g-1}(t)d\mu$$

$$= \int_{X_1} f(g(t))\phi_g(t)d\mu$$

$$= \int_{X_2} f(t)d\mu \ .$$

Therefore, for all $f \in L^\infty[X_2]$,

$$\int_{X_2} f(t)[1 - \phi_g(g^{-1}(t))\phi_{g-1}(t)]d\mu = 0$$

and iii) has been established.

Let e be the function on X_2 defined by $e(x) = 1$ for all x in X_2. Then

$$\int_{X_1} \phi_g(t)d\mu = \int_{X_1} e(g(t))\phi_g(t)d\mu$$

$$= \int_{X_2} e(t)d\mu$$

$$= \mu(X_2) \ .$$

Definition 2.2. Let μ and γ be two positive measures on a measure space E and let F be a measurable subset of E. Then μ and γ are said to be equivalent relative to F, denoted by $\mu \sim \gamma$ rel F, provided that for all measurable sets $A \subset E$, $\mu(F \cap A) = 0$ if and only if $\gamma(F \cap A) = 0$.

Let Λ^n_j be the collection of all subsets of $\{1,2,\cdots,n\}$ with j elements in each subset.

<u>Lemma 2.2.</u> Let E be a measure space with $\mu_1,\cdots,\mu_n$ positive measures defined on it. Then there exists a decomposition of E into Borel sets of the following form:

i) $E = \bigcup_{i=1}^{n} E_i$, $E_i \cap E_j = \emptyset$ for $i \neq j$;

ii) $E_i = \bigcup_{\sigma \in \Lambda^n_i} E_i^\sigma$, $E_i^\sigma \cap E_i^\beta = \emptyset$ for $\sigma \neq \beta$;

iii) if $m,n \in \sigma$, then $\mu_m \sim \mu_n$ rel E_i^σ; and

iv) if $k \notin \sigma$, then $\mu_k(E_i^\sigma) = 0$.

<u>Proof</u>: The proof will be by induction on the number of measures defined on E.

Suppose μ_1 and μ_2 are two positive measures on

E. Then by the Lebesque decomposition theorem there exists a pair of measures, μ_1^a and μ_1^s, such that:

1) $\mu_1 = \mu_1^a + \mu_1^s$;

2) $\mu_1^a << \mu_2$; and

3) $\mu_1^a \perp \mu_1^s$, and $\mu_2 \perp \mu_1^s$.

Let $E_1^{\{1\}}$ be a Borel subset of E such that for any measurable subset A of E, $\mu_1^s(A) = \mu_1^s(A \cap E_1^{\{1\}})$, $\mu_2(E_1^{\{1\}}) = 0$, and $\mu_1^a(E_1^{\{1\}}) = 0$. Now considering μ_1^a and μ_2 as measures on $E \setminus E_1^{\{1\}}$, there exists a pair of measures, μ_2^a and μ_2^s, such that on $E \setminus E_1^{\{1\}}$ we have:

4) $\mu_2 = \mu_2^a + \mu_2^s$;

5) $\mu_2^a << \mu_1^a$; and

6) $\mu_2^a \perp \mu_2^s$, and $\mu_1^a \perp \mu_2^s$.

Let $E_1^{\{2\}}$ be a Borel subset of $E \setminus E_1^{\{1\}}$, such that for any measurable subset A of $E_1 \setminus E_1^{\{1\}}$; $\mu_2^s(A) = \mu_2^s(A \cap E_1^{\{2\}})$, $\mu_2^a(E_1^{\{2\}}) = 0$, and $\mu_1^a(E_1^{\{2\}}) = 0$. Since μ_1^s

is concentrated on $E_1^{\{1\}}$, it follows that

$$\mu_1(E_1^{\{2\}}) = \mu_1^a(E_1^{\{2\}}) + \mu_1^s(E_1^{\{2\}}) = 0.$$

Put $E_2^{\{1,2\}} = E \setminus (E_1^{\{1\}} \cup E_2^{\{2\}})$. By 2), on $E_2^{\{1,2\}}$ we have that $\mu_1^a << \mu_2$. However, μ_1^s is concentrated on $E_1^{\{1\}}$, hence $\mu_1 << \mu_2$ on $E_2^{\{1,2\}}$. By 5), on $E_2^{\{1,2\}}$ we have $\mu_2^a << \mu_1^a$. But again by the definition of $E_1^{\{1\}}$ and $E_1^{\{2\}}$, it is also true that $\mu_2 << \mu_1$ on $E_2^{\{1,2\}}$. Thus we have shown that $\mu_2 \sim \mu_1$ rel $E_2^{\{1,2\}}$, and that the sets $E_1^{\{1\}}$, $E_1^{\{2\}}$, and $E_2^{\{1,2\}}$ satisfy the conditions of the lemma.

Now let $\mu_1, \cdots, \mu_n$ be n positive measures defined on E. For $\mu_1, \cdots, \mu_{n-1}$, by the induction hypothesis, there exists the following decomposition of E:

7) $E = \bigcup_{i=1}^{n-1} F_i$, $F_i \cap F_j = \emptyset$ for $i \neq j$;

8) $F_i = \bigcup_{\sigma \in \Lambda_i^{n-1}} F_i^\sigma$, $F_i^\sigma \cap F_i^\beta = \emptyset$ for $\sigma \neq \beta$

9) if $m, n \in \sigma$, then $\mu_m \sim \mu_n$ rel F_i^σ; and

10) if $k \notin \sigma$, then $\mu_k(E_i^\sigma) = 0$.

Let $\sigma \in \Lambda_{i}^{n-1}$ and $k \in \sigma$. Considering μ_k and μ_n as measures on F_i^σ, there exist measures μ_n^s and μ_n^a such that:

11) $\mu_n = \mu_n^a + \mu_n^s$;

12) $\mu_n^a << \mu_k$; and

13) $\mu_n^a \perp \mu_n^s$, and $\mu_n^s \perp \mu_k$.

Let K_j^σ be a Borel subset of F_i^σ, such that for any measurable subset A of F_i^σ ; $\mu_n^s(A) = \mu_n^s(A \cap K_j^\sigma)$, $\mu_n^a(K_j^\sigma) = 0$, and $\mu_k(K_j^\sigma) = 0$. Now considering μ_n^a and μ_k as positive measures on $F_j^\sigma \setminus K_j^\sigma$, there exists a pair of measures, μ_k^a and μ_k^s , on $F_j^\sigma \setminus K_j^\sigma$ such that:

14) $\mu_k = \mu_k^a + \mu_k^s$;

15) $\mu_k^a << \mu_n^a$;

16) $\mu_k^s \perp \mu_k^a$; and $\mu_k^s \perp \mu_n^a$.

For $\sigma \subset \{1,2,\cdots,n-1\}$ and $1 \leq j \leq n-1$, let E_j^σ be a Borel subset of $F_j^\sigma \setminus K_j^\sigma$ such that for any measurable subset A of $F_j^\sigma \setminus K_j^\sigma$, $\mu_k^s(A) = \mu_k^s(A \cap E_j^\sigma)$,

$\mu_k^s(E_j^\sigma) = 0$, and $\mu_n^a(E_j^\sigma) = 0$. Since μ_n^s is concentrated on K_j^σ and $E_j^\sigma \subset F_j^\sigma \setminus K_j^\sigma$, it follows that $\mu_n(E_j^\sigma) = 0$. Since $E_j^\sigma \subset F_j^\sigma$, $m,n \in \sigma$ implies $\mu_m \sim \mu_n$ rel E_j^σ and also $p \notin \sigma$ implies $\mu_p(E_j^\sigma) = 0$. Thus for $\sigma \subset \{1,2,\cdots,n-1\}$ and $1 \leq j \leq n-1$, E_j^σ satisfies iii).

For $\sigma \in \Lambda_j^{n-1}$, let $E_{j+1}^{\sigma\cup\{n\}} = F_j^\sigma \setminus (E_j^\sigma \cup K_j^\sigma)$. If $k \in \sigma$, then by 12), 15), and the definitions of E_j^σ and K_j^σ, it follows that $\mu_n \sim \mu_k$ rel $E_{j+1}^{\sigma\cup\{n\}}$. Again since $E_{j+1}^{\sigma\cup\{n\}} \subset F_j^\sigma$ we have that $E_j^{\sigma\cup\{n\}}$ satisfies iii).

From the definition of K_j^σ we observe that for $k \in \sigma$, $\mu_k(K_j^\sigma) = 0$. For $k \notin \sigma$ it is also true that $\mu_k(K_j^\sigma) = 0$, since $K_j^\sigma \subset F_j^\sigma$. Thus if we let $E_1^{\{n\}} = \bigcup_{j=1}^{n-1} \left(\bigcup_{\sigma\in\Lambda_j^{n-1}} K_j^\sigma \right)$, it follows that $\mu_k(E_1^{\{n\}}) = 0$ for $1 \leq k \leq n-1$.

Observing that $F_j^\sigma = E_{j+1}^{\sigma\cup\{n\}} \cup E_j^\sigma \cup K_j^\sigma$ and that they are pairwise disjoint, we have that $\{E_j^\sigma\}$, $\sigma \in \Lambda_j^n$, $1 \leq j \leq n$, satisfy the conditions of this lemma.

Definition 2.3. Suppose f is a continuous real valued function on $[a,b]$. For each real number y, let $M_f(y)$ be the number (finite or infinite) of points, X, in $[a,b]$ at which $f(x) = y$. Call M_f the multiplicity function of f.

It is known that M_f is a Borel function and that $\int M_f(y)dy$ is the total variation of f on $[a,b]$. For a discussion of the above see [5] or [8].

The next lemma is one of a technical nature but is quite important.

Lemma 2.3. Suppose that f is a continuous real valued function on $[0,1]$ such that $\bigcup_{1 \leq i < \infty} f^{-1}(M_f^{-1}(i))$ has Lebesque measure equal to 1. Then there exists a collection of sets $\{E_{i,j}\}$ $1 \leq j \leq i$, $i = 0,1,2,\cdots$, that satisfy the following:

i) $E_{i,j} \cap E_{k,\ell} = \emptyset$ if $i \neq k$ and $j \neq \ell$;

ii) $f(E_{i,j}) = f(E_{i,k})$ for $1 \leq j,k \leq i$;

iii) f restricted to $E_{i,j}$ is one to one;

iv) $\mu(f^{-1}(E) \cap E_{i,j}) = 0$ if and only if $\mu(f^{-1}(E) \cap E_{i,k}) = 0$, where μ is Lebesque measure, E is a Borel subset in $[0,1]$, and $1 \leq j,k \leq i$; and

v) $\mu(\bigcup_{k=1}^{j} E_{j,k} \cap E) = \mu(f^{-1}(f(E_{j,k_0})) \cap E)$

for measurable sets E such that $A_{\Pi_E} \in (\{A_f\}')'$.

<u>Proof</u>: For each natural number i, let $F_i = M_f^{-1}(i)$ and $K_i = f^{-1}(F_i)$. Define inductively the collection $\{K_{i,j}\}$, $1 \leq j \leq i$, $i = 1,2,\cdots$ as follows;

$$K_{i,1} = \min\{f^{-1}(y) | y \in F_i\},$$

and for $1 < j \leq i$,

$$K_{i,j} = \min\{f^{-1}(y) \setminus \bigcup_{k=1}^{j-1} K_{i,k} | y \in F_i\}.$$

We will prove that the collection $\{K_{i,j}\}$ has the following properties:

1) $f(K_{i,j}) = f(K_i) = F_i$ for $1 \leq j \leq i$, $i = 1,2,\cdots$

2) $K_{i,j} \cap K_{i,\ell} = \emptyset$ for $j \neq \ell$;

3) $\bigcup_{j=1}^{i} K_{i,j} = K_i$;

4) f restricted to $K_{i,j}$ is one to one; and

5) each $K_{i,j}$ is measurable.

First it can be noted that $x \in K_{i,j}$ if and only if there exist i real numbers, $x_1 < x_2 < \cdots < x_i$ such that $f(x_k) = f(x)$ for $k = 1,2,\cdots,i$, and x is equal to the jth element in this ordering.

From their definition, $K_{i,j} \subset K_i$ and thus $f(K_{i,j}) \subset f(K_i)$. Now suppose $y \in f(K_i)$. Then

$$y \in f(f^{-1}(K_i)) = F_i = M_f^{-1}(i) \quad \text{and}$$

hence, there exist real numbers, $x_1, x_2, \cdots, x_i$, such that $f(x_k) = y$, $k = 1,2,\cdots,i$. If $x_1 < x_2 < \cdots < x_i$, then by the above note, $x_j \in K_{i,j}$. Therefore, $y \in f(K_{i,j})$ and it follows that $f(K_{i,j}) = f(K_i) = F_i$.

For $j \neq \ell$, $K_{i,j}$ and $K_{i,\ell}$ are clearly disjoint. If $x \in K_i$, then $f(x) \in F_i$ and hence $f^{-1}(f(x))$ contains i real numbers. Suppose that x is the jth element in $f^{-1}(f(x))$. Then $x \in K_{i,j}$ and since $K_{i,j} \subset K_i$, we have that $K_i = \bigcup_{j=1}^{i} K_{i,j}$.

Now suppose $f(x) = f(z)$ for $x,z \in K_{i,j}$. But this implies that both x and z are the j'th elements in $f^{-1}(f(x))$. Therefore x must be equal to z and f is one to one on $K_{i,j}$.

For each positive integer n, let $I_n^k = (\frac{2^{k-1}}{2^n}, \frac{2^k}{2^n})$ $k = 1,2,\cdots,n$ and $J_n^k = I_n^k \cap K_i$. For each positive integer n, let D_n be the union of those J_n^k such that $J_n^k \cap K_{i,j} \neq \emptyset$, and let $D = \bigcap_{n=1}^{\infty} D_n$. We will show that $D_n \subset K_{i,j}$. Let $x \in D$ and suppose that $x \notin K_{i,j}$. Since $D \subset K_i$, there exist real numbers, $x_1,x_2,\cdots,x_i$, such that $x_1 < x_2 < \cdots < x_i$ and $f(x_i) = f(x)$. By 3), there exists an integer ℓ such that $x \in K_{i,\ell}$. Assume that $\ell < i$. Since $x \in D$, by the nesting property of the J_n^k there exists a sequence $\{y_n\}$ of distinct elements in $K_{i,j}$ that converge to x. We may assume without loss of generality that this sequence has the additional properties that for all n, $f(y_n) < f(x)$, $y_{n+1} < y_n$, and $x < y_n$. Since $x_n \in K_{i,j}$, for $n = 1,2,\cdots$, we have the associated infinite sets $\{y_n^k\}$ where $f(y_n^k) = f(y_n)$, $y_n^j = y_n$, and $y_n^{k-1} < y_n^k$ for $k = 1,2,\cdots,i$. We may assume that each $\{y_n^k\}$ converges, for if not, we could pick convergent subsequences in an orderly fashion and obtain a subsequence of $\{y_n\}$ which has the desired properties. Since $f(y_n^k) = f(y_n)$ for $n = 1,\cdots,i$ each $\{y_n^k\}$ must converge to a element of $\{x_k\}_{k=1}^{i}$. But $\{x_n^j\}$ converges to $x = x_\ell$ where $\ell < j$, hence there exists a least integer $p > \ell$ such that none of the sequences $\{y_n^k\}$ converges to x_p.

Let n_0 be a positive integer such that $\{y_{n_0}^k\}$ for $k = 1,\cdots,i$, is within $\min\{|x_k - x_{k-1}|\}_{k=2}^{i}$ from the point to which they converge. Suppose that k_0 is such that the sequence $\{y_n^{k_0}\}$ converges to x^{p-1}. Now $f(y_{n_0}^{k_0}) < f(x) = f(y_{p_0})$, hence by the intermediate value theorem there exists a z between $y_{n_0}^{k_0}$ and y_p such that $f(z) = f(y_{n_0+1}^{k_0})$. Now by the choice of n_0,

$z \neq x^k_{n_0+1}$ for $k = 1,2,\cdots,i$. Hence $M_f(f(y^k_{n_0+1})) \geq i + 1$, which is a clear contradiction since $y^k_{n_0+1} \in K_{i,j} \subset K_i$. Therefore $x \in K_{i,j}$ and $D \subset K_{i,j}$.

Now if $x \in K_{i,j}$ and x is not one of the dyadic rationals, then for each positive integer n, $x \in J^k_n$ for some k. Hence, $x \in D_n$ for all positive integers n and thus $x \in D$. Now D is clearly a measurable set and if we add those dyadic rationals to D which are contained in $K_{i,j}$, the resulting set will still be measurable. Therefore $K_{i,j}$ is measurable.

For E a Borel subset of F_i, $1 \leq j \leq i$, let

6) $\mu_{i,j}(E) = \mu(f^{-1}(E) \cap K_{i,j})$

where μ is Lebesque measure on $[0,1]$. For each i, by Lemma 2.2, there exists the following decomposition of F_i:

7) $F_i = \bigcup_{j=1}^{i} F_{i,j}$, $F_{i,j} \cap F_{i,k} = \emptyset$ for $j \neq k$;

8) $F_{i,j} = \bigcup_{\sigma \in \Lambda^i_j} F^\sigma_{i,j}$, $F^\sigma_{i,j} \cap F^\beta_{i,j} = \emptyset$ for $\sigma \neq \beta$;

9) if $m, n \in \sigma$, then $\mu_{i,m} \sim \mu_{i,n}$ rel $F^{\sigma}_{i,j}$; and

10) if $k \notin \sigma$, then $\mu_{i,k}(F^{\sigma}_{i,j}) = 0$.

For $1 \leq j, m \leq i$, and $\sigma \in \Lambda^{i}_{j}$, put

11) $E^{\sigma}_{i,j,m} = f^{-1}(F^{\sigma}_{i,j}) \cap K_{i,m}$.

By σ_j, $1 \leq j \leq i$, we shall mean the jth element in σ, where we consider the elements in σ to be linearly ordered.

Finally, for $1 \leq j \leq i$, $i = 1,2,\cdots$, let

12) $E_{i,j} = \bigcup_{k} \bigcup_{\sigma \in \Lambda^{k}_{j}} E^{\sigma}_{k,i,\sigma_j}$.

We will show that the collection $\{E_{i,j}\}$, $1 \leq j \leq i$, $i = 1,2,\cdots$, satisfies the conditions of the lemma.

From 11) it follows that,

$$E^{\sigma}_{\ell,j,n} \cap E^{\sigma'}_{\ell',j',n'}$$

$$= (f^{-1}(F^{\sigma}_{i,j}) \cap K_{\ell,n}) \cap (f^{-1}(F^{\sigma'}_{i',j'}) \cap K_{\ell',n'})$$

$$= f^{-1}(F^{\sigma}_{i,j} \cap F^{\sigma'}_{i',j'}) \cap (K_{\ell,n} \cap K_{\ell',n'}).$$

The above, together with 2) and 8), implies that $E^{\sigma}_{\ell,j,n} \cap E^{\sigma}_{\ell',j',n'} = \emptyset$ if $\sigma \neq \sigma'$, $j \neq j'$, $\ell \neq \ell'$ and $n \neq n'$. However, $E^{\sigma}_{\ell,i,n} \subset E_{m,k}$ if and only if $i = m$ and n is the k th element in σ. Hence, $E_{i,j} \cap E_{m,k} = \emptyset$, for $i \neq m$ and $j \neq k$.

From 1), 3) and 11), it follows that

$$\begin{aligned} f(E^{\sigma}_{\ell,i,n}) &= f(f^{-1}(F^{\sigma}_{\ell,i}) \cap K_{\ell,n}) \\ &\subset F^{\sigma}_{\ell,i} \cap f(K_{\ell,n}) \\ &= F^{\sigma}_{\ell,i} \cap F_{\ell} = F^{\sigma}_{\ell,i} \ . \end{aligned}$$

Suppose that $y \in F^{\sigma}_{\ell,i} \subset F_{\ell}$. Then by 1) there exists $x \in K_{\ell,n}$ such that $f(x) = y$. Therefore, $x \in f^{-1}(F^{\sigma}_{\ell,i}) \cap K_{\ell,n} = E^{\sigma}_{\ell,i,n}$ and $y \in f(E^{\sigma}_{\ell,i,n})$ which implies that $F^{\sigma}_{\ell,i} \subset f(E^{\sigma}_{\ell,\ell,n})$. Hence, we have established that

13) $f(E^{\sigma}_{\ell,i,n}) = F^{\sigma}_{\ell,i}$.

Now from 12) and 13) we see that for $1 \leq j,n \leq i$,

$$f(E_{i,j}) = \bigcup_{k} \bigcup_{\sigma\in\Lambda^{k}_{j}} f(E^{\sigma}_{k,i,\sigma_j})$$

$$= \bigcup_{k} \bigcup_{\sigma \in \Lambda_j^k} f(E^{\sigma}_{k,i,\sigma_n})$$

$$= f(E_{i,n}).$$

Suppose $f(x) = f(y)$ and $x,y \in E_{i,j}$. Then there exist $\sigma, \sigma', \ell, \ell', n$ and n' such that $x \in E^{\sigma}_{\ell,i,n}$ and $y \in E^{\sigma'}_{\ell',i',n'}$. From 13) we have that $f(x) \in F_{\ell}$ and $f(y) \in F_{\ell'}$, hence $\ell = \ell'$. $x \in E^{\sigma}_{\ell,\ell',n}$ implies that $f(x) \in F^{\sigma}_{\ell,i}$, thus $f(y) \in F^{\sigma}_{\ell,i}$ and $\sigma = \sigma'$. Moreover, since $x \in E^{\sigma}_{\ell,i,n}$, $y \in E^{\sigma}_{\ell,i,n'}$ and $x,y \in E_{i,j}$, we must have that both $E^{\sigma}_{\ell,i,n}$ and $E^{\sigma}_{\ell,i,n'}$ are subsets of $E_{i,j}$, but this can only happen if both n and n' are the j th elements in σ or equivalently $n = n'$. We have now shown that $x,y \in E^{\sigma}_{\ell,i,n} \subset K_{\ell,m}$. However, f is one to one on $K_{\ell,m}$, so that $x = y$ and thus f restricted to $E_{i,j}$ is one to one.

Suppose E is a Borel subset of $F^{\sigma}_{\ell,i}$. From 9) it follows that $\mu_{i,\sigma_j}(E) = 0$ if and only if $\mu_{i,\sigma_k}(E) = 0$. However, 6) implies that this is equivalent to $\mu(f^{-1}(E) \cap K_{i,\sigma_j}) = 0$ if and only if

$\mu(f^{-1}(E) \cap K_{i,\sigma_k}) = 0$. Since $E^{\sigma}_{\ell,i,\sigma_j} = f^{-1}(F^{\sigma}_{\ell,i}) \cap K_{\ell,\sigma_j}$ and $E \subset F^{\sigma}_{\ell,i}$, we have that

$$\mu(f^{-1}(E) \cap K_{i,\sigma_j}) = \mu(f^{-1}(E) \cap E^{\sigma}_{\ell,i,\sigma_j})$$

$$= \mu(f^{-1}(E) \cap E_{i,j}).$$

Similarly, $\mu(f^{-1}(E) \cap K_{i},\sigma_k) = \mu(f^{-1}(E) \cap E_{i,k})$ and thus, $\mu(f^{-1}(E) \cap E_{i,k}) = 0$ if and only if $\mu(f^{-1}(E) \cap E_{i,j}) = 0$, where E is a Borel subset of $F^{\sigma}_{\ell,i}$. For E a Borel subset of the range of f, $E = \bigcup_{\ell} \bigcup_{\sigma \in \Lambda^{\ell}_{i}} (F^{\sigma}_{\ell,i} \cap E)$ and thus is the union of pairwise disjoint sets of the form for which iv) has been shown to hold.

From 12) and 13) it follows that

$$f^{-1}(f(E_{j,k_0})) = \bigcup_{i} \bigcup_{\sigma \in \Lambda^{i}_{j}} f^{-1}(f(E^{\sigma}_{i,j,\sigma_{k_0}}))$$

$$= \bigcup_{i} \bigcup_{\sigma \in \Lambda^{i}_{j}} f^{-1}(F^{\sigma}_{i,j})$$

$$= \bigcup_i \bigcup_{k=1}^{i} \bigcup_{\sigma \in \Lambda_j^i} (f^{-1}(F_{i,j}^{\sigma}) \cap K_{i,k})$$

$$= \bigcup_i \bigcup_{k=1}^{i} \bigcup_{\sigma \in \Lambda_j^i} E_{i,j,k}^{\sigma} ,$$

but $\bigcup_{k=1}^{j} E_{j,k} = \bigcup_{k=1}^{j} \bigcup_i \bigcup_{\sigma \in \Lambda_j^i} E_{i,j,\sigma_k}^{\sigma}$ and hence

$$f^{-1}(f(E_{j,k_0})) = \bigcup_{k=1}^{j} E_{j,k} \cup (\bigcup_i \bigcup_{\sigma \in \Lambda_j^i} \bigcup_{m \notin \sigma} E_{i,j,m}^{\sigma}).$$

For $A_{\pi_E} \in (\{A_f\}')'$, or equivalently, $E = f^{-1}(f(E))$

and $m \notin \sigma$, we have that

$$\mu(E \cap E_{i,j,m}^{\sigma}) = \mu(f^{-1}(f(E)) \cap [f^{-1}(F_{i,j}^{\sigma}) \cap K_{i,n}])$$

$$= \mu(f^{-1}(f(E) \cap F_{i,j}^{\sigma}) \cap K_{i,m})$$

$$= \mu_{i,m}(f(E) \cap F_{i,j}^{\sigma})$$

$$\leq \mu_{i,m}(F_{i,j}^{\sigma}) = 0.$$

Finally we have

$$\mu(f^{-1}(f(E_{k,k_0})) =$$

$$\mu\Big(\bigcup_{k=1}^{j} E_{j,k} \cap E\Big) + \sum_{i} \sum_{\sigma \in \Lambda} \frac{i}{j} \sum_{m \notin \sigma} \mu(E \cap E^{\sigma}_{i,j,m})$$

$$= \mu\Big(\bigcup_{k=1}^{j} E_{j,k} \cap E\Big).$$

As we will see later the $\{E_{i,j}\}$ corresponds, in the finite case, to a partitioning of eigensubspaces according to the multiplicity of their eigenvalues.

For $1 \leq i,j \leq n$, $n = 1,2,\cdots,$ let $f|_{E_{n,i}}$ be the restriction of f to $E_{n,i}$, and define ${}_n g_{i,j}$ from $E_{n,i}$ onto $E_{n,j}$ by

$${}_n g_{i,j}(x) = f^{-1}_{|E_{n,i}}\big(f|_{E_{n,j}}(x)\big).$$

Suppose E is a Borel subset of $E_{n,i}$. Then from Lemma 2.2, $\mu(f^{-1}(f(E)) \cap E_{n,i}) = 0$ if and only if $\mu(f^{-1}(f(E)) \cap E_{n,j}) = 0$, but $\mu({}_n g_{i,j}(E)) = \mu(f^{-1}_{|E_{n,j}}(f(E))) = \mu(f^{-1}(f(E)) \cap E_{n,j})$, and $\mu(E) = \mu(f^{-1}_{|E_{n,i}}(f(E))) = \mu(f^{-1}(f(E)) \cap E_{n,i})$. Hence for E

a Borel subset of $E_{n,i}$, $\mu(E) = 0$ if and only if $\mu({}_n g_{i,j}(E)) = 0$. Therefore, for $1 \leq i,j \leq n$, $n = 1,2,\cdots$, each ${}_n g_{i,j}$ satisfies the hypothesis of Lemma 2.1, so there exists a $\emptyset_{{}_n g_{i,j}}$ corresponding to each ${}_n g_{i,j}$. For $n = 0,1,2,\cdots$, let

$$g_k(x) = \begin{cases} \sum_{n=k}^{\infty} \sum_{i=1}^{n} \emptyset_{{}_n g_{k,i}}(x) & \text{for } x \in \bigcup_{k=i}^{\infty} E_{k,i}, \\ 0 & \text{otherwise}. \end{cases}$$

<u>Theorem 2.3.</u> Let f be as in Lemma 2.3 and let g_k be as defined above. Then $\{L^2[\bigcup_{k=i}^{\infty} E_{k,i}], g_i\}$ is the canonical decomposition system for the weakly closed ring $(\{A_f\}')'$.

<u>Proof</u>: Let e be the function on $[0,1]$ defined by $e(x) = 1$ for all x in $[0,1]$. Since $L^{\infty}[0,1] \subset (\{A_f\}')'$, it follows that e is a cyclic vector for $\{A_f\}'$. Therefore, there exists a measure $\hat{\mu}$ on $\mathfrak{M}_f$, the maximal ideal space of $(\{A_f\}')'$, such that for $A_g \in (\{A_f\}')'$,

$$(A_g e, e) = \int_{m_f} g^{\wedge}(m)\, d\hat{\mu}(m) .$$

Suppose $A_h \in (\{A_f\}')'$. Then

$$(A_h g_k, g_k) = \int_0^1 h(t) \Big[\sum_{n=k}^{\infty} \sum_{i=1}^{n} \emptyset_{{}_n g_{k,i}}(t) \Big] d\mu$$

$$= \sum_{n=k}^{\infty} \sum_{i=1}^{n} \int_{E_{n,k}} h(t)\, \emptyset_{{}_n g_{k,i}}(t)\, d\mu .$$

From Lemma 2.1, we have that

$$\sum_{n=k}^{\infty} \sum_{i=1}^{n} \int_{E_{n,k}} h(t)\, \emptyset_{{}_n g_{k,i}}(t)\, d\mu =$$

$$\sum_{n=k}^{\infty} \sum_{i=1}^{n} \int_{E_{n,i}} h({}_n g_{i,k}(t))\, \emptyset_{{}_n g_{k,i}}({}_n g_{i,k}(t))\, \emptyset_{{}_n g_{i,k}}(t)\, d\mu$$

$$= \sum_{n=k}^{\infty} \sum_{i=1}^{n} \int_{E_{n,i}} h({}_n g_{i,k}(t))\, d\mu .$$

Now by Theorem 1.2 there exists a bounded Borel function h' on the range of f, such that $h = h' \circ f$. Therefore, for $t \in E_{n,i}$, we have that

$$h({}_n g_{i,k}(t)) = h'(f({}_n g_{i,k}(t)))$$
$$= h'(f(f^{-1}|_{E_{n,k}}(f(t))))$$
$$= h'(f(t)) = h(t) .$$

Thus,

$$\sum_{n=k}^{\infty} \sum_{i=1}^{n} \int_{E_{n,i}} h({}_n g_{i,k}(t))\,d\mu = \sum_{n=k}^{\infty} \sum_{i=1}^{n} \int_{E_{n,i}} h(t)\,d\mu$$
$$= \int_{\bigcup_{n=k}^{\infty} \bigcup_{i=1}^{n} E_{n,i}} h(t) \quad d\mu .$$

Combining the above results we have that

$$(A_h g_k, g_k) = \int_{\bigcup_{n=k}^{\infty} \bigcup_{i=1}^{n} E_{n,i}} h(t) \quad d\mu .$$

For $i = 1,2,\cdots,$ let

$$S_i = f^{-1}(f(\bigcup_{k=i}^{\infty} E_{k,i})) .$$

If $A_{\Pi_E} \in (\{A_f\}')'$, then from the definition of $\hat{\mu}$ and Theorem 2.2, it follows that

$$\mu(E) = \int_0^1 \Pi_E(t)d\mu = (A_{\Pi_E} e, e) = \int_{m_f} \hat{A}_{\Pi_E}(m)d\hat{\mu}$$

$$= \int_{m_f} \Pi_{\hat{E}}(m)d\hat{\mu} = \hat{\mu}(\hat{E}).$$

Using the above connection between μ and $\hat{\mu}$, combined with Lemma 2.3 part 5), we have that

$$\int_{\bigcup_{n=k}^{\infty} \bigcup_{i=1}^{n} E_{n,i}} \Pi_E(t)d\mu = \mu\left(\bigcup_{n=k}^{\infty} \bigcup_{i=1}^{n} E_{n,i} \cap E\right)$$

$$= \sum_{n=k}^{\infty} \mu\left(\bigcup_{i=1}^{n} E_{n,i} \cap E\right)$$

$$= \sum_{n=k}^{\infty} \mu(f^{-1}(f(E_{n,k})) \cap E)$$

$$= \mu(S_i \cap E)$$

$$= \hat{\mu}(\hat{S}_i \cap E)$$

$$= \int_{m_f} \Pi_{\hat{S}_k}(m)\Pi_{\hat{E}}(m)d\hat{\mu} .$$

For $A_h \in (\{A_f\}')'$, approximating with simple functions and using the above, we have established that

$$(A_h g_k, g_k) = \int_{\bigcup_{n=k}^{\infty} \bigcup_{i=1} E_{n,i}} h(t)d\mu = \int_{m_f} \hat{A}_h(m)\Pi_{\hat{g}_k}(m)d\hat{\mu}.$$

Now we shall prove that $(\{A_f\}')' g_k = L^2[\bigcup_{i=k}^{\infty} E_{i,k}]$, where we consider $L^2[\bigcup_{i=k}^{\infty} E_{i,k}]$ as a subspace of $L^2[0,1]$ in the natural manner. We will first show that for $k = 1,2,\cdots,$ $1/g_k \in L^{\infty}[\bigcup_{i=k}^{\infty} E_{i,k}]$. Let $E = \{x:\emptyset_{{}_n g_{i,j}}(x) < 0\}$. From Lemma 2.1 it follows that

$$0 \geq \int_E \emptyset_{{}_n g_{i,j}}(t)d\mu = \int \Pi_{{}_n g_{i,j}^{-1}(E)}({}_n g_{i,j}(t))\emptyset_{{}_n g_{i,j}}(t)d\mu$$

$$= \int \Pi_{{}_n g^{-1}{}_{i,j}(E)}(t)d\mu \geq 0,$$

and hence, $\mu(E) = 0$ or equivalently $\emptyset_{{}_n g_{i,j}} \geq 0$ a.e. Since ${}_n g_{k,k}(x) = x$ for $x \in E_{n,k}$, we have that $\emptyset_{{}_n g_{k,k}}(x) = 1$ and thus, $\sum_{i=1}^{n} \emptyset_{{}_n g_{k,i}}(x) \geq 1$ for $x \in E_{n,k}$. Therefore, $g_k(x) \geq 1$ for $x \in \bigcup_{i=k}^{\infty} E_{i,k}$

and we have established that $\frac{1}{g_k} \in L^\infty[\bigcup_{i=k}^{\infty} E_{i,k}]$.

Now, for each bounded Borel function g on $[0,1]$ such that $g(x) = 0$ for $x \notin \bigcup_{i=k} E_{i,k}$, we show that there exists a function $\tilde{g}$ with the property that

$A_{\tilde{g}} \in (\{A_f\}')'$ and $\tilde{g}(x) = g(x)$ for $x \in \bigcup_{i=k}^{\infty} E_{i,k}$.

Define the function g' on the range of f as follows,

$$g'(y) = \begin{cases} g(f|_{E_{i,k}}^{-1}(y)) & \text{if } y \in f(E_{i,k}) \text{ for some } i \geq k, \\ 0 & \text{otherwise.} \end{cases}$$

From 13) in the proof of Lemma 2.3, we see that $f(E_{i,k}) \cap f(E_{j,k}) = \emptyset$ if $i \neq j$, and hence g' is well defined. Also we note that for $x \in E_{i,k}$, $i \geq k$,

$$g(x) = g(f|_{E_{i,k}}^{-1}(f(x))) = g'(f(x)).$$

Now let $\tilde{g} = g' \circ f$. Then from the above statement, $\tilde{g}|_{E_{i,k}} = g$ for $i \geq k$. By Theorem 1.2 we know that

$A_{\tilde{g}} \in (\{A_f\}')'$.

Now if h is a continuous function in $L^2[\bigcup_{i=k}^{\infty} E_{i,k}]$, then there exists a function $\tilde{h}$ such that

$A_{\tilde{h}} \widetilde{\frac{1}{g_k}} \in (\{A_f\}')'$. Since

$$A_{\tilde{h}} \widetilde{\frac{1}{g_k}} g_k = \tilde{h} \widetilde{\frac{1}{g_k}} g_k = h,$$

we have that, $h \in (\{A_f\}')' g_k$, from which it follows that $(\{A_f\}')' g_k = L^2[\bigcup_{i=k}^{\infty} E_{i,k}]$.

All that remains to be shown is that $\mu([0,1]) = \mu(\bigcup_{k=1}^{\infty} \bigcup_{i=k}^{\infty} E_{i,k})$. From the construction in Lemma 2.3, we can partition $[0,1]$ as follows

$$\begin{aligned}
[0,1] &= f^{-1}(\bigcup_{i=1}^{\infty} F_i) \cap \bigcup_{i=1}^{\infty} K_i \\
&= \bigcup_{i=1}^{\infty} [f^{-1}(\bigcup_{j=1}^{i} \bigcup_{\sigma \in \Lambda} {}_i F_{i,j}^{\sigma}) \cap \bigcup_{m=1}^{i} K_{i,m}] \\
&= \bigcup_{i=1}^{\infty} \bigcup_{j=1}^{i} \bigcup_{m=1}^{i} \bigcup_{\sigma \in \Lambda_j^i} f^{-1}(F_{i,j}^{\sigma}) \cap K_{i,m} \\
&= \bigcup_{i=1}^{\infty} \bigcup_{j=1}^{i} \bigcup_{m=1}^{i} \bigcup_{\sigma \in \Lambda_j^i} E_{i,j,m}^{\sigma} .
\end{aligned}$$

We showed in Lemma 2.3 that $E_{i,j,m}^{\sigma} \subset E_{p,q}$ for some p and q if and only if $m \notin \sigma$ for some σ. If $m \notin \sigma$, then

$$\mu(E^{\sigma}_{i,j,m}) = \mu(f^{-1}(F^{\sigma}_{i,j}) \cap K_{i,m}) = \mu_{i,m}(F^{\sigma}_{i,j}) = 0.$$

Therefore, $\mu([0,1]) = \mu(\bigcup_{i=1}^{\infty} \bigcup_{j=1}^{i} E_{i,j})$ and since $\sum_{k=1}^{\infty} (\{A_f\}')'g_k = \sum_{k=1}^{\infty} L^2[\bigcup_{i=k}^{\infty} E_{i,k}]$, the proof of the theorem is completed.

<u>Remark 2.2.</u> If, in Lemma 2.3 and Theorem 2.3, we replace $[0,1]$ by $X_n = \{1,2,\cdots,n\}$, Lebesque measure by counting measure and consider the ring $(\{A_f\}')'$, where f is a real valued continuous function on X_n, then the calculation of a canonical decomposition system is greatly simplified. The sets F_i defined in the proof of Lemma 2.3, are easily seen to be the set of eigenvalues of A_f whose eigensubspaces have dimension i. For each i, K_i is equal to $f^{-1}(F_i)$ and $\{K_{i,j} \mid 1 \leq j \leq i\}$ is again a partition of K_i with the properties 1) $f(K_{i,j}) = f(K_i)$, 2) $K_{i,j} \cap K_{i,\ell} = \emptyset$ if $j \neq \ell$, and 3) $\bigcup_{j=1}^{i} K_{i,j} = K_i$. It is easily seen

that $K_{i,j}$ satisfy all the conditions of Lemma 2.3 and hence $E_{i,j} = K_{i,j}$. For finite dimensional Hilbert spaces, the weak operator topology agrees with the norm operator topology. Thus the $\hat{S}_i$ in Theorem 2.3, which correspond to the S_{n_i} in Definition 2.1 of the canonical decomposition, are the following

$$S_{n_i} = \hat{S}_i = f^{-1}(f(\bigcup_{k=i}^{\infty} E_{k,i})) = \bigcup_{k=i}^{\infty} F_k,$$

where all but a finite number of the F_k are empty. Therefore, we see that the canonical decomposition gives us a natural generalization of the ordering of eigenvalues according to the dimension of the eigensubspaces. Hence the canonical decomposition is very well suited for the unitary equivalence problem.

CHAPTER III

In this chapter we want to determine necessary and sufficient conditions for two normal operators to be unitarily equivalent. We will use the canonical decomposition system to reduce the problem to one in which the weakly closed ring generated by each normal operator has a cyclic vector.

We begain the study of unitary equivalence by considering some examples. Again we will use $L^{\infty}[0,1]$ as a ring of operators on the Hilbert space $L^{2}[0,1]$.

Let S be the middle third Cantor set and let $c(x)$ be the Cantor function; that is, the function on $[0,1]$ with the following properties:

i) $c([0,1]) = [0,1]$;

ii) c is monotone increasing;

iii) c is continuous; and

iv) $c'(x) = 0$ for $x \notin S$.

Let $f(x) = x + c(x)$ and $g(x) = 2x$. We first note that A_f and A_g have the same spectrum, and both generate $L^\infty[0,1]$ in the weak operator topology. If $P(\lambda)$ and $Q(\lambda)$ are the spectral resolutions of A_f and A_g respectively, then

$$P(f(S)) = A_{\Pi_{f^{-1}(f(S))}} = A_{\Pi_S} \text{ and } Q(f(S)) = A_{\Pi_{g^{-1}(f(S))}} .$$

However the measure of the cantor set S is zero and the measure of $g^{-1}(f(S))$ is 1/2. Therefore, there does not exist a unitary operator U such that $U^*A_{\Pi_S}U = A_{\Pi_{g^{-1}(f(S))}}$, and it follows that A_f and A_g are not unitarily equivalent.

The above example shows that neither the comparison of the spectrum, nor the comparison of the rings which the operators generate, is a sufficient condition for determining when two normal operators are unitarily equivalent. There seems to be a relevant connection with regard to the function taking sets of measure zero to sets of positive measure. The next example will give us more insight into this connection.

For p and q positive real numbers, let $f_p(x) = x^p$, let ${}_ph_q(x) = x^{p/q}$ and let ${}_pU_q$ be the operator on $L^2[0,1]$ defined by

$$({}_pU_qk)(t) = k({}_ph_q(t))({}_ph'_q(t))^{1/2}$$
$$= \left(\frac{p}{q}\right)^{1/2} k(t^{p/q})\, t^{\frac{p-q}{2q}} .$$

For $h,\ k \in L^2[0,1]$, we have that

$$({}_pU_qk, {}_pU_qk) = \int_0^1 |k(t^{p/q})|^2 p/q\ t^{\frac{p-q}{q}}\, dt$$
$$= \int_0^1 |k(x)|^2 dx$$
$$= (k,k), \text{ and}$$
$$({}_pU_qk, h) = \int_0^1 k(t^{p/q})\ \overline{h(t)}\ \left(\frac{p}{q}\right)^{1/2} t^{\frac{p-q}{2q}}\, dt$$
$$= \int_0^1 k(x)\ \overline{h(x^{q/p})}\left(\frac{q}{p}\right)^{1/2} (x^{q/p})^{\frac{p-q}{2q}}\ x^{\frac{q-p}{2p}}\, dx$$
$$= \int_0^1 k(x)\ \overline{h(x^{q/p})}\left(\frac{q}{p}\right)^{1/2} x^{\frac{q-p}{2p}}\, dx$$
$$= (k, {}_qU_ph).$$

Thus we see that ${}_pU_q$ is an isometry and ${}_qU_p$ is its adjoint. Moreover,

$$({}_pU^*_q\ {}_pU_q)(t) = k(t)\left(\frac{p}{q}\,\frac{q}{p}\right)^{1/2} t^{\frac{p-q}{2p}}\, t^{\frac{q-p}{2p}}$$

$$= k(t).$$

Similarly, ${}_pU_q\ {}_pU^*_q = I$, and thus ${}_pU_q$ is a unitary operator. It is easily verified that ${}_pU_qA_{f_q} = A_{f_p}{}_pU_q$.

It should be noted that in this example the function ${}_ph_q$ has the property that $f_p(t) = f_q({}_ph_q(t))$ and that it is a continuous function which takes sets of measure zero to sets of measure zero. Hence, for E a Borel subset of real numbers, $\mu(f_p{}^{-1}(E)) = 0$ if and only if $\mu(f_q{}^{-1}(E)) = 0$, where μ is Lebesgue measure on $[0,1]$.

The above examples point the way to the following theorem.

Theorem 3.1. Let f and g be continuous, monotone increasing, real valued functions on $[0,1]$. Then A_f is unitarily equivalent to A_g if and only if $\mu(f^{-1}(E)) = 0$ is equivalent to $\mu(g^{-1}(E)) = 0$, for each Borel subset E of real numbers.

Proof: Suppose that $\mu(f^{-1}(E)) = 0$ if and only if $\mu(g^{-1}(E)) = 0$, for each Borel set E. Let k be the continuous function from $[0,1]$ onto $[0,1]$ defined by

$$k(x) = g^{-1}(f(x)) \text{ for } x \in [0,1].$$

If E is a Borel subset of $[0,1]$, then

$$\mu(k(E)) = \mu(g^{-1}(f(E))) = 0$$

is equivalent to

$$\mu(E) = \mu(f^{-1}(f(E))) = 0.$$

Hence we have shown that $\mu(E) = 0$ if and only if $\mu(k(E)) = 0$, for E a Borel subset of $[0,1]$. Therefore, k satisfies the hypothesis of Lemma 2.1 so that there exists a function ϕ_k such that

1) $\displaystyle\int_0^1 f(k(t))\phi_k(t)d\mu = \int_0^1 f(t)d\mu$, and

2) $\phi_k(k^{-1}(t))\phi_{k^{-1}}(t) = 1$ a.e.

Define the operator U on $L^2[0,1]$ as follows, for $h \in L^2[0,1]$, let

$$(Uh)(t) = h(k(t))(\phi_k(t))^{1/2}.$$

As in the example, property 1) implies that U is an isometry and property 2) implies that the operator U^*, defined by

$$(U^*h)(t) = h(k^{-1}(t))(\phi_{k^{-1}}(t))^{1/2}$$

for $h \in L^2[0,1]$, is the adjoint and inverse of U. Thus, U is a unitary operator on $L^2[0,1]$, and moreover, for $h \in L^2[0,1]$, we have that

$$\begin{aligned}(UA_g h)(t) &= (Ugh)t \\ &= g(k(t))h(k(t))(\phi_k(t))^{1/2} \\ &= f(t)h(k(t))(\phi_k(t))^{1/2} \\ &= (A_f Uh)(t).\end{aligned}$$

Therefore, A_f is unitarily equivalent to A_g.

Now let us assume that there exists a unitary operator U such that $A_f U = UA_g$.

Let $\mathcal{B}$ be the collection of all Borel subsets of $[0,1]$ and let $\mathcal{N}$ be the collection of all null sets in $\mathcal{B}$. By $\mathcal{B}/\mathcal{N}$ we mean the collection of all equivalence classes of sets in $\mathcal{B}$, where two sets E and F are equivalent if and only if $E \Delta F = (E\backslash F) \cup (F\backslash E) \in \mathcal{N}$. We let $\tilde{E}$ denote the equivalence class which contains E.

Now if $E \in \mathcal{B}$, then $U^*A_{\pi_E}U$ is a projection in $(\{A_g\}')' = L^\infty[0,1]$. Therefore, there exists a Borel set F such that $U^*A_{\pi_E}U = A_{\pi_F}$. We use this to define a function Φ, from $\mathcal{B}$ to $\mathcal{B}/\mathcal{N}$, by $\Phi(E) = \tilde{F}$, where $UA_{\pi_E}U^* = A_{\pi_F}$. Now suppose that $E_1, E_2 \in \mathcal{B}$, $F_1 \in \Phi(E_1)$, $F_2 \in \Phi(E_2)$ and $E_1 \cap E_2 = \emptyset$. Then $A_{\pi_{F_1 \cap F_2}} = A_{\pi_{F_1}}A_{\pi_{F_2}} = (UA_{\pi_{E_1}}U^*)(UA_{\pi_{E_2}}U^*) = UA_{\pi_{E_1}}A_{\pi_{E_2}}U^* = UA_{\pi_{E_1 \cap E_2}}U^* = 0$. Thus, $F_1 \cap F_2 \in \mathcal{N}$, but $F_1 \cap F_2 \in \mathcal{N}$ implies that $A_{\pi_{F_1}} + A_{\pi_{F_2}} = A_{\pi_{F_1 \cup F_2}}$. Therefore, $UA_{\pi_{E_1 \cup E_2}}U^* = UA_{\pi_{E_1}}U^* + UA_{\pi_{E_2}}U^* = A_{\pi_{F_1}} + A_{\pi_{F_2}} = A_{\pi_{F_1 \cup F_2}}$, or equivalently, $\Phi(E_1 \cup E_2) = \widetilde{F_1 \cup F_2} = \tilde{F}_1 \cup \tilde{F}_2 = \Phi(E_1) \cup \Phi(E_2)$. Let $E \in \mathcal{B}$ and $F \in \Phi(E)$. Then $UA_{\pi_{[0,1]\setminus E}}U^* = U(A_{\pi_{[0,1]}} - A_{\pi_E})U^* = A_{\pi_{[0,1]}} - A_{\pi_F} = A_{\pi_{[0,1]\setminus F}}$, so that $\Phi([0,1]\setminus E) = \widetilde{[0,1]}\setminus\Phi(E)$. Let e be the constant function 1 on $[0,1]$ and $h = Ue$. If $F \in \Phi(E)$, then

$$U\Pi_E = UA_{\Pi_E}e = A_{\Pi_F}Ue = \Pi_F h.$$

Now suppose that $E = \bigcup_{i=1}^{\infty} E_i$, $E_i \in \mathcal{B}$, $E_i \cap E_j = \emptyset$ if $i \neq j$, and $F_i \in \Phi(E_i)$. Then, in the $L^2[0,1]$ norm,

$$\Pi_E = \lim_{n \to \infty} \sum_{i=1}^{n} \Pi_{E_i}.$$

Therefore,

$$U\Pi_E = \lim_{n \to \infty} \sum_{i=1}^{n} U\Pi_{E_i}$$

$$= \lim_{n \to \infty} \sum_{i=1}^{n} \Pi_{F_i} h$$

$$= \Pi_{\bigcup_{i=1}^{\infty} F_i} h,$$

or equivalently, $\Phi(E) = \bigcup_{i=1}^{\infty} \Phi(E_i)$. Hence, Φ is a σ-homomorphism from $\mathcal{B}$ onto $\mathcal{B}/\mathcal{N}$.

Let $A_1 = [0,1]$, and for each rational number α in $[0,1]$, let $A_\alpha \in \Phi([0,\alpha])$. If $\alpha < \beta$, then $\Phi([0,\alpha]) \subseteq \Phi([0,\beta])$.

For $\alpha < \beta$, let $E_{\alpha,\beta} = A_\alpha \setminus A_\beta$ and let $E = \bigcup_{\alpha<\beta} E_{\alpha,\beta}$. Since $\mu(E) = 0$, if we let $B_\alpha = A_\alpha \cup E$, then $B_\alpha \in \Phi([0,\alpha])$ and $\alpha < \beta$ implies that $B_\alpha \subset B_\beta$. Define φ from $[0,1]$ into $[0,1]$ by the following, for each $x \in [0,1]$, let

$$\varphi(x) = \inf\{\alpha | x \in B_\alpha\}.$$

It follows immediately that

$$\{x | \varphi(x) \leq t\} = \bigcup_{\alpha \leq t} B_\alpha,$$

and hence that φ is measurable. From the definition of φ it is also easily seen that $\varphi^{-1}[0,\alpha] = B_\alpha \in \Phi[0,\alpha]$. Therefore, since φ^{-1} and Φ are σ-homomorphisms, $\varphi^{-1}(E) \in \Phi(E)$ for all Borel sets E. Moreover, since $\varphi^{-1}(E) \in \Phi(E)$, we have that

$$UA_{\Pi_E}U^* = A_{\Pi_{\varphi^{-1}(E)}}$$

and thus, $\varphi^{-1}(E)$ has measure zero if and only if E has measure zero, for each Borel subset E of $[0,1]$. Let $S(t) = \sum_{i=1}^{n} a_i \Pi_{E_i}$ where $E_i \in \mathcal{B}$ and $E_i \cap E_j = \emptyset$ if $j \neq j$.

Then

$$(US)(t) = \sum_{i=1}^{n} a_i(U\Pi_{E_i})(t)$$

$$= \sum_{i=1}^{n} a_i \Pi_{\varphi^{-1}(E_i)}(t)h(t)$$

$$= \sum_{i=1}^{n} a_i \Pi_{E_i}(\varphi(t))h(t)$$

$$= S(\varphi(t))h(t),$$

and since the simple functions are dense in $L^2[0,1]$, for $k \in L^2[0,1]$ we have that

$$(Uk)(t) = k(\varphi(t))h(t).$$

Similarly, for U^* there exist a φ^* and a h^* such that for $k \in L^2[0,1]$,

$$(U^*k)(t) = k(\varphi^*(t))h^*(t).$$

Since $U^*U = I$, it follows that

$$h(\varphi^*(t))h(t) = e(\varphi(\varphi^*(t)))h(\varphi^*(t))h(t)$$

$$= (U^*Ue)(t) = 1,$$

and if $i(x) = x$, then

$$\varphi(\varphi^*(x)) = \varphi(\varphi^*(x))h(\varphi^*(x))h^*(x)$$

$$= (U^*Ui)(x) = x.$$

Similarly, $\varphi^* \circ \varphi = i$, and thus, $\varphi^* = \varphi^{-1}$. From our assumption we know that $UA_f = A_g U$, and by applying e to both sides of the equality, we have that

$$\begin{aligned} f(\varphi(t))h(t) &= (UA_f e)(t) \\ &= (A_g Ue)(t) \\ &= g(t)h(t). \end{aligned}$$

Since $h(t) \neq 0$ a.e., $f(\varphi(t)) = g(t)$. Moreover, φ and φ^{-1} take sets of measure zero to sets of measure zero, so that $\mu(f^{-1}(E)) = 0$ if and only if $\mu(g^{-1}(E)) = 0$ for each Borel subset E of $[0,1]$.

Let $P(\lambda)$ and $Q(\lambda)$ be the spectral resolutions of A_f and A_g respectively. From Lemma 1.1 we know that, for a Borel set E, $P(E) = A_{\Pi_{f^{-1}(E)}}$ and $Q(E) = A_{\Pi_{g^{-1}(E)}}$. Therefore, we have the following corollary.

<u>Corollary 3.1</u>. Suppose that f and g are continuous, monotone increasing, real valued functions on $[0,1]$. Let $P(\lambda)$ and $Q(\lambda)$ be the spectral resolutions of A_f and A_g respectively. Then A_f is unitarily equivalent to A_g

if and only if $P(E) = 0$ is equivalent to $Q(E) = 0$, for each Borel subset E of real numbers.

Definition 3.1. Let f be a continuous, monotone increasing, real valued function on $[0,1]$ and let $P(\lambda)$ be its spectral resolution. Then μ_f will be used to denote the Borel measure defined as follows. For each Borel subset E of real numbers, let $\mu_f(E) = (P(E)e,e)$ where e is the constant function 1.

It is easily seen that if $P(E) = 0$ then $\mu_f(E) = 0$. Assume that $\mu_f(E) = 0$ for a Borel set E. Then

$$||P(E)e||^2 = (P(E)e,P(E)e) = (P(E)e,e) = \mu_f(E) = 0.$$

Now let F be any other Borel set. Since e is a cyclic vector for $(\{A_f\}')'$ and $P(E)[P(F)e] = P(F)[P(E)e] = 0$, we have that $P(E) = 0$. Therefore, we have shown that $P(E) = 0$ if and only if $\mu_f(E) = 0$. This, together with Corollary 3.2, gives us the following.

Corollary 3.2. Let f,g, μ_f and μ_g be as in Definition 3.1. Then A_f is unitarily equivalent to A_g if and only if μ_f and μ_g are mutually absolutely continuous.

Corollary 3.2 will be the model for determining if two normal operators on a separable Hilbert space are unitarily equivalent.

<u>Definition 3.2</u>. Let N be a normal operator on the Hilbert space H, and $P(\lambda)$ its spectral resolution. Then for $\eta \in H$, let μ^N_η denote the Borel measure defined in the following way. For each Borel set E, let $\mu^N_\eta(E) = (P(E)\eta,\eta)$.

<u>Lemma 3.1</u>. Let N be a normal operator on the Hilbert space H, and let M be the weakly closed ring generated by N and N^*. If ξ and η are both cyclic vectors for the ring M, then μ^N_ξ and μ^N_η are mutually absolutely continuous.

Proof: Let $P(\lambda)$ be the spectral resolution for N, and suppose E is a Borel subset of complex numbers. Since $\mu^N_\eta(E) = (P(E)\eta,\eta) = (P(E)\eta,P(E)\eta) = ||P(E)||^2$, $\mu^N_\eta(E) = 0$

if and only if $P(E)\eta = 0$. For F, Borel, we have that $P(E)[P(F)\eta] = P(F)[P(E)\eta] = 0$, but η is a cyclic vector for M so that, the linear span of $\{P(F)\eta | F$ is a Borel set$\}$ is dense in H. Therefore $P(E)\eta = 0$ if and only if $P(E) = 0$. Thus, we have shown that $\mu^N_\eta(E) = 0$ if and only if $P(E) = 0$. If, in the above, we replace η by ξ, we arrive at the same result, and the conclusion of the lemma follows immediately.

The next lemma shows us that if $(\{N\}')'$ has a cyclic vector, then we are not far removed from the conditions of Theorem 3.1.

Lemma 3.2. Let N be a normal operator on the Hilbert space H, and let M be the weakly closed ring generated by N and N^*. If M has a cyclic vector η, then there exists an isometric operator U from H onto $L^2[\mu^N_\eta]$ such that $U^{-1}NU = A_i$, where i is the function defined by $i(x) = x$ and $A_i f = if$ for all $f \in L^2[\mu^N_\eta]$.

Proof: For E a Borel set, let $U\Pi_E = P(E)\eta$. Extend U linearly to the collection of all simple functions.

Thus, U now maps a dense subset of H linearly onto a dense subset of $L^2[\mu^N_\eta]$. Moreover,

$$\begin{aligned}||\Pi_E||^2 &= \int \Pi_E d\mu^N_\eta \\ &= \mu^N_\eta(E) \\ &= (P(E)\eta,\eta) \\ &= ||U\Pi_E||.\end{aligned}$$

Therefore, U can be extended to an isometry from $L^2[\mu^N_\eta]$ onto H.

If E and F are Borel sets, then

$$\begin{aligned}(NU\Pi_E, P(F)\eta) &= \int \lambda\, d(P(\lambda)U\Pi_E, P(F)\eta) \\ &= \int \lambda\, d(P(\lambda)P(E)\eta, P(F)\eta) \\ &= \int \lambda \Pi_{E\cap F}(\lambda)\, d(P(\lambda)\eta,\eta) \\ &= (\Pi_E i, \Pi_F) \\ &= (U(\Pi_E i), P(F)\eta).\end{aligned}$$

Since the linear span of elements of the form $P(F)\eta$ is dense in H, we have that $NU\Pi_E = U(i\Pi_E)$, for all Borel sets E. If $S = \sum_{k=1}^{n} a_k \Pi_{E_k}$, then

$$\begin{aligned} NUS &= NU\left(\sum_{k=1}^{n} a_k \Pi_{E_k}\right) \\ &= \sum_{k=1}^{n} a_k \, NU\Pi_{E_k} \\ &= \sum_{k=1}^{n} a_k Ui\Pi_{E_k} \\ &= U(iS). \end{aligned}$$

As the simple functions are dense in $L^2[\mu^N_\eta]$, we can conclude that for all $f \in L^2[\mu^N_\eta]$,

$$NUf = U(if) = UA_i f.$$

We are now ready to state necessary and sufficient conditions for two normal operators to be unitarily equivalent in the case where the weakly closed rings that they generate have cyclic vectors.

Theorem 3.2. Suppose H is a separable Hilbert space, $N_k (k = 1,2)$ is a normal operator in $B(H)$ and M^k is the weakly closed symmetric ring generated by N_k and N^*_k. If ξ_k is a unit cyclic vector for M_k and if $\mu^{N_k}_{\xi_k}$ is the Borel measure which corresponds to N_k and ξ_k, then N_1 is unitarily equivalent to N_2 if and only if $\mu^{N_1}_{\xi_1}$ and $\mu^{N_2}_{\xi_2}$ are mutually absolutely continuous.

Proof: Let $P_k(\lambda)$ be the spectral resolution of $N_k (k = 1,2)$. Assume that there exists a unitary operator U such that $UN_1 = N_2U$. For E a Borel set, we have that

$$\begin{aligned}\mu^{N_2}_{\xi_2}(E) &= (P_2(E)\xi_2, \xi_2)\\ &= (U^*P_1(E)U\xi_2, \xi_2)\\ &= (P_1(E)U\xi_2, U\xi_2)\\ &= \mu^{N_1}_{U\xi_2}(E).\end{aligned}$$

Since $M^1U\xi_2 = UM^2U^*U\xi_2 = UM^2\xi_2 = H$, $U\xi_2$ is a cyclic vector for M^1. Therefore, by Lemma 3.1, $\mu^{N_1}_{\xi_1}$ and $\mu^{N_1}_{U\xi_2}$ are mutually absolutely continuous. Thus, $\mu^{N_2}_{\xi_2} = \mu^{N_1}_{U\xi_2}$ implies that $\mu^{N_1}_{\xi_1}$ and $\mu^{N_2}_{\xi_2}$ are mutually absolutely continuous.

Now assume that $\mu^{N_1}_{\xi_1}$ and $\mu^{N_2}_{\xi_2}$ are mutually absolutely continuous. Then by the Radon-Nikodym Theorem, there exists a non-negative real valued function h such that

$$\int f d\mu^{N_1}_{\xi_1} = \int fh d\mu^{N_2}_{\xi_2}.$$

Define V from $L^2[\mu^{N_1}_{\xi_1}]$ onto $L^2[\mu^{N_2}_{\xi_2}]$ by

$$Vf = f(h)^{1/2} \text{ for each } f \in L^2[\mu^{N_1}_{\xi_1}].$$

In the same manner as before it is easily seen that V is an isometry. Let $V_k (k = 1,2)$ be the isometry from $L^2[\mu^{N_k}_{\xi_k}]$ onto H given by Lemma 3.2. Moreover, let U be the operator from H onto H defined by

$$U = V_1 V V_2^{-1}.$$

Again let i be the function given by $i(x) = x$, for x a complex number and let $A^k_i (k = 1,2)$ be the operator such that for $f \in L^2[\mu^{N_k}_{\xi_k}]$, $A^k_i f = if$. From Lemma 3.2 we know that $V_k^{-1} N V_k = A^k_i (k = 1,2)$ and clearly,

$A^1_iV = VA^2_i$. Therefore, $V_1^{-1}N_1V_1V = VV_2^{-1}N_2V_2$, or equivalently, $N_1V_1VV_2^{-1} = V_1VV_2^{-1}N_2$. Since $U = V_1VV_2^{-1}$ we have that $N_1U = UN_2$. Since U is the product of isometries,it is also an isometry and since

$$(U\eta, \xi) = (V_1VV_2^{-1}\eta, \xi) = (\eta, V_2V^{-1}V_1^{-1}\xi),$$

$U^* = V_2V^{-1}V_1^{-1}$. Therefore,

$$UU^* = (V_1VV_2^{-1})(V_2V^{-1}V_1^{-1}) = I, \text{ and}$$

$$U^*U = (V_2V^{-1}V_1^{-1})(V_1VV_2^{-1}) = I.$$

Hence, U is the required unitary operator.

Before considering the case where the weakly closed rings generated by the normal operators do not have cyclic vectors, we show that we can replace the measure theoretic condition of Theorem 3.2 with a topological condition on the maximal ideal spaces of their weakly closed rings.

Let N be a normal operator on a separable Hilbert space H, M the weakly closed ring generated by N and N^*, and ξ a cyclic vector for M. Suppose also that μ^N_ξ and A_i are as in Lemma 3.2 and that $\mathfrak{M}_{A_i}$ is the maximal ideal space of the weakly closed ring generated by A_i and A^*_i. We use the characterization and notation of Theorem 2.2 for $\mathfrak{M}_{A_i}$. Let t be the restriction map from $\mathfrak{M}_{A_i}$ to the spectrum of N; that is, for $\emptyset \in \mathfrak{M}_{A_i}$, $t(\emptyset) = x$ where $\emptyset(A_f) = f(x)$ for f a continuous function on the spectrum of N.

Lemma 3.3. Let N, A_i, $\mathfrak{M}_{A_i}$, μ^N_ξ and t be as above. If F is a closed set of complex numbers such that $\mu^N_\xi(F) > 0$ and U an open set which contains F, then $\hat{F} \subset t^{-1}(U)$.

Proof: In the proof of Lemma 3.1, we showed that $\mu^N_\xi(F) > 0$ if and only if $P(F) \neq 0$. Since A_{Π_F} is the projection in M_{A_i} which corresponds to $P(F)$, we have that $A_{\Pi_F} \neq 0$ so that $\hat{F}$ is an open set in $\mathfrak{M}_{A_i}$. There exists a continuous function f such that f is equal to one on F, zero on the complement of U and between one and zero everywhere else. Now

$$A_{\Pi_F} \leq A_f \leq A_{\Pi_U},$$

which implies that

$$\hat{\Pi_F} \leq \hat{A_f} \leq \hat{A_{\Pi_U}} .$$

Suppose that $\emptyset \in \hat{F}$. Then

$$1 = \hat{\Pi_F}(\emptyset) = \hat{A_f}(\emptyset) = f(t(\emptyset)) \leq 1.$$

Thus, $f(t(\emptyset)) = 1$ and we have that $t(\emptyset) \in U$ or equivalently $\emptyset \in t^{-1}(U)$.

The next two lemmas begin to develop a relationship between the topology on the maximal ideal space and the measure induced from the cyclic vector for the ring.

Lemma 3.4. Let A_i, $\mathcal{M}_{A_i}$, $\mu^N{}_\xi$ and t be as in Lemma 3.3. If F is a closed set of complex numbers such that $\mu^N{}_\xi(F) > 0$, then $\hat{F} \subset t^{-1}(F)$.

Proof: Suppose $\hat{F} \not\subset t^{-1}(F)$. Then there exists a $\emptyset$ such that $\emptyset \in \hat{F}$ and $\emptyset \notin t^{-1}(F)$. Since $t(\emptyset) \notin F$, there exist open sets $O_\emptyset$ and O_F with the following properties: 1) $t(\emptyset) \in O_F$; 2) $F \subset O_F$; and 3) $O_\emptyset \cap O_F$ is empty. From Lemma 3.3 we have that $\hat{F} \subset t^{-1}(O_F)$ and hence, it follows that $\emptyset \in t^{-1}(O_F)$. However, we now have that

$$\emptyset \in t^{-1}(O_F) \cap t^{-1}(O_\emptyset) = t^{-1}(O_F \cap O_\emptyset),$$

which is a clear contradiction. Therefore, we have established that $\hat{F} \subset t^{-1}(F)$.

Lemma 3.5. Again, let A_i, $\mathfrak{M}_{A_i}$, $\mu^N{}_\xi$ and t be as in Lemma 3.3. For each Borel set E, $\hat{A}_i^{-1}(E)$ has void interior if and only if $\mu^N{}_\xi(E) = 0$.

Proof: First we note that since i is continuous, $\hat{A}_i(\emptyset) = \emptyset(A_i) = t(\emptyset)$, for $\emptyset \in \mathfrak{M}_{A_i}$.

Assume that the interior of $\hat{A}_i^{-1}(E)$ is not empty. Then by Theorem 2.2 there exists a Borel set F such that $\mu^N{}_\xi(F) \neq 0$ and $\hat{F} \subset \hat{A}_i^{-1}(E)$. It now follows that

$$t^{-1}(F) = \hat{A}_i^{-1}(F) \subset \hat{A}_i^{-1}(E) = t^{-1}(E),$$

and hence $F \subset E$. However, $\mu^N_\xi(F) > 0$ implies that $\mu^N_\xi(E) > 0$.

Now assume that $\mu^N_\xi(E) > 0$. From Theorem 1 in section 39 of [2], we know that measures of the form μ^N_ξ are regular. Therefore, there exists a closed set F with positive measure contained in E. From Lemma 3.4 we have that

$$\hat{F} \subset t^{-1}(F) = \hat{A}_i^{-1}(F) \subset \hat{A}_i^{-1}(E).$$

Therefore, the interior of $\hat{A}_i^{-1}(E)$ is not empty.

Now, using Lemma 3.2, we transfer the conditions of Lemma 3.5 over to N and $\mathfrak{M}$.

Theorem 3.3. Suppose that H is a separable Hilbert space, N is a normal operator in $B(H)$, M is the weakly closed symmetric ring generated by N and N^*, ξ is a unit cyclic vector for M, and μ^N_ξ is the Borel measure which corresponds to N and ξ. Then, if E is a Borel subset of the complex plane, $\hat{N}^{-1}(E)$ has void interior if and only if $\mu^N_\xi(E) = 0$.

Proof: Let A_i, M_{A_i} be as before, and let V be the isometry from H onto $L^2[\mu^N{}_\xi]$ given in Lemma 3.2. The map $N \to V^{-1}NV$ defines a homeomorphism φ from $\mathfrak{M}$ onto $\mathfrak{M}_{A_i}$ such that $N^\wedge(\emptyset) = A_i^\wedge(\varphi(\emptyset))$ for each $\emptyset \in \mathfrak{M}$. Now the theorem follows directly from Lemma 3.5.

Combining Theorem 3.2 and Theorem 3.3 we arrive at the following theorem of Porcelli and Butts which is given in [1].

Theorem 3.4. Suppose H is a separable Hilbert space, $N_i (i = 1,2)$ is a normal operator in $B(H)$, M^i is the weakly closed symmetric ring generated by N_i and $N^*{}_i$, ξ_i is a unit cyclic vector for M^i and $\mathfrak{M}_i$ is the maximal ideal space of M^i. Then a necessary and sufficient condition for N_1 to be unitarily equivalent to N_2, is that for each Borel set E of complex numbers $N_1^{\wedge -1}(E)$ has void interior if and only if $N_2^{\wedge -1}(E)$ has void interior.

We will now consider the case where the weakly closed rings generated by the normal operators do not have cyclic vectors.

<u>Definition 3.3</u> Let H be a separable Hilbert space, let N be a normal operator in $B(H)$, and let $\{(K_i, \eta_i)\}$ be a canonical decomposition for the weakly closed symmetric ring, M, generated by N and N^*. For each i, let $\mu^N{}_i$ be the measure $\mu^N{}_{\eta_i}$ defined in Definition 3.2. The collection $\{\mu^N{}_i\}$ is called the multiplicity set for N.

<u>Remark 3.1.</u> It is clear that $K_i, M|_{K_i}, \eta_i$ and S_{η_i}, the maximal ideal space of $M|_{K_i}$, satisfy the hypothesis of Theorem 3.3. Thus, for each Borel set E, $\mu^N{}_i(E) = 0$ if and only if $N^{\wedge^{-1}}(E) \cap S_{\eta_i}$, has void interior. Therefore, if $S_{\eta_i} = S_{\eta_j}$, then $\mu^N{}_i$ and $\mu^N{}_j$ are mutually absolutely continuous.

Finally, we will now use the multiplicity set of the normal operators to reduce the general case to that of Theorem 3.2.

Theorem 3.5. Suppose H is a separable Hilbert space, $N_i (i = 1,2)$ is a normal operator on H, M^k is the weakly closed symmetric ring generated by N_i and N^*_i, and $\mathfrak{m}_i$ is the maximal ideal space of M^k. Moreover, if $\{\mu^{N_k}_j\} (k = 1,2)$ is the multiplicity set for N_k, then N_1 and N_2 are unitarily equivalent if and only if for each i, $\mu^{N_1}_i$ and $\mu^{N_2}_i$ are mutually absolutely continuous.

Proof: Suppose that U is a unitary operator such that $UN_1 = N_2U$. Let $A' \in (M^1)'$ and $B \in M^2$. Then $U^*BU \in M^1$ and $U^*BUA' = A'U^*BU$, which implies that $B(UA'U^*) = (UA'U^*)B$. Thus, we have that $U^*(M^2)'U \subset (M^1)'$. Similarly $U(M^1)'U^* \subset (M^2)'$ and hence $U(M^1)'U^* = (M^2)'$. Also, if ξ_0 is a cyclic vector for $(M^1)'$, then $(M^1)'\xi_0 = H = U(M^1)'U^*U\xi_0 = (M^2)'U\xi_0$ and therefore $U\xi_0$ is a cyclic vector for $(M^2)'$.

Suppose $\{(K_i,\eta_i)\}$ is a canonical decomposition for M^1. We claim that $\{(L_i,U\eta_i)\}$ where $L_i = M^2U\eta_i$ is a canonical decomposition for M^2. Since $H = \Sigma_i \oplus UM^1U^*U\eta_i = \Sigma_i \oplus M^2U\eta_i$, all that we must show is that $\varphi_{U\eta_i} = \Pi_{S_{U\eta_i}}$. If $A \in M^2$, then

$$\begin{aligned}
(AU\eta_i, U\eta_i) &= (U^*AU\eta_i, \eta_i) \\
&= \int_{S_{\eta_i}} (U^*AU)^\wedge(m)d\mu_1 \\
&= (P_{S_{\eta_i}} U^*AU\xi_0, \xi_0) \\
&= (UP_{S_{\eta_i}} U^*AU\xi_0, U\xi_0) \\
&= \int_{\mathcal{M}_2} (UP_{S_{\eta_i}} U^*)^\wedge(m)A^\wedge(m)d\mu_2,
\end{aligned}$$

and $(AU\eta_i, U\eta_i) = \int_{\mathcal{M}_2} A^\wedge(m)\varphi_{U\eta_i}(m)d\mu_2(m)$. Therefore,

$(UP_{S_{\eta_i}} U^*)^\wedge = \varphi_{U\eta_i}$ a.e., and since $UP_{S_{\eta_i}} U^*$ is a projection in M^2, we have that $\varphi_{U\eta_i} = \Pi_{S_{U\eta_i}}$. Now if $P(\lambda)$ and $Q(\lambda)$ are the spectral resolution of N_1 and N_2 respectively, then for E a Borel set, $\mu^{N_1}_{\eta_i}(E) =$

$(P(E)\eta_i, \eta_i) = (U^*Q(E)U\eta_i, \eta_i) = (Q(E)U\eta_i, U\eta_i) =$

$\mu^{N_2}_{U\eta_i}(E)$. Since by Lemma 3.1 the choice of cyclic vectors for K_i and L_i does not affect the absolute continuity of the measures, we have that $\mu^{N_1}_i$ and $\mu^{N_2}_i$ are mutually absolutely continuous.

We shall now prove the sufficiency part of the theorem. Let $\{(K_i, \eta_i)\}$ and $\{(L_i, \xi_i)\}$ be canonical decomposition systems for M^1 and M^2 respectively. Since $M^1|_{K_i}$ ($M^2|_{L_i}$) has $\eta_i(\xi_i)$ as a cyclic vector, and $S_{\eta_i}(S_{\xi_i})$ as its maximal ideal space, Theorem 3.2 can be applied to $\mu^{N_1}_i$ and $\mu^{N_2}_i$. Therefore, there exists a unitary operator U_i such that $U_iK_i = L_i$ and $U_iN_1 = N_2U_i$. If $U = \Sigma_i U_i$, then U is a unitary operator and $UN_1 = N_2U$.

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